A FV Scheme for Maxwell's equations

Convergence Analysis on unstructured meshes

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ABSTRACT. *In [10], Remaki developped a new Finite Volume Scheme for the resolution of the heterogeneous Maxwell equations in three dimensions. The scheme is based on a leapfrog time discretization and involves a centered flux formula for discretization in space. Numerical tests have shown the performance of the method, namely on unstructured grids. In this paper, we will present some recent convergence results in L^2 in the case of unstructured grids which satisfy an appropriate "aspect condition". This condition will be dicussed and illustrated by some numerical results.*

KEYWORDS: *Maxwell's equations, finite volume scheme, centered flux, stability, convergence*

1. Introduction

In [10], M. Remaki introduced a new Finite Volume Scheme for the resolution of the heterogeneous time-dependent Maxwell equations in three dimensions. The scheme is based on a leapfrog time discretization and involves a centered flux formula for discretization in space. From a numerical point of view, it gathers many advantages. It turns out to be non diffusive and is thus well adapted to the simulation of propagation phenomena. The analysis of the scheme has been carried out on cartesian grids in [10]. It has been proved that the scheme is stable (under an appropriate CFL condition) and second-order accurate in space and in time. Moreover, the dispersion of the scheme has been shown to be of second order (and thus of the same order as the classical FDTD-scheme of Yee [11]).

Finite volume methods have been studied widely in the context of nonlinear scalar conservation laws [6, 7, 3]. In the general case of nonlinear systems, few results are known. Kröner et al. proved in [8] that the L^2_{loc}-limit of a convergent sequence of Finite Volume approximations may be identified with a weak solution of the problem. In [2], a FV Scheme for symetric hyperbolic systems is analyzed and the error in L^2 is shown to be of order $1/2$ with respect to the mesh size.

In electromagnetism, finite volume methods became popular in the early '90ies. Those involving primal and dual grids suffer essentially from the lack of three dimensional mesh generators. Other methods are based on higher-order-upwind discretization in space [1, 5], known as MUSCL schemes. The main drawback of these methods is its numerical diffusion which does affect seriously the accuracy of the solution in long-run computations.

Finite volume schemes in electromagnetism derive from the Maxwell equations written in conservative form as a linear hyperbolic system. However, few theoretic results are known, especially for convergence. Recently, Piperno et al. [4] investigated stability results for the scheme in the case of the heterogeneous Maxwell equations with metallic and absorbing boundary conditions.

2. Setting of the electromagnetic problem

The electromagnetic problem we are interested in, is the following: given two incident fields $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$, we are looking for $(\boldsymbol{H}, \boldsymbol{E})$ satisfying

$$\left\{ \begin{array}{lcl} \mu \partial_t \boldsymbol{H} + \mathrm{curl}\,(\boldsymbol{E}) & = & 0 \ \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \varepsilon \partial_t \boldsymbol{E} - \mathrm{curl}\,(\boldsymbol{H}) & = & 0 \ \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \end{array} \right. , \tag{1}$$

where $\varepsilon(x)$ and $\mu(x)$ are piecewise constant functions representing respectively the electric permittivity and magnetic permeability of the medium. We assume that there are constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha \leq \varepsilon(x), \mu(x) \leq \beta \quad \text{for almost every } x \in \mathbb{R}^3. \tag{2}$$

A functional frame for problem (1) has been given in [9].

3. Discretization

In this section we introduce the discrete formulation of problem (1) with the help of a discrete curl operator. We will show that the scheme actually is a Finite Volume Scheme with a centered numerical flux.

For a given mesh $\mathcal{T} = (T_j)_{j\in\mathbb{Z}}$ of $\mathbb{R}^3$, made up by a finite number of connected polyhedral finite volumes, we introduce a discrete curl operator acting on sequences $(\boldsymbol{U}_j)_{j\in\mathbb{Z}})$ of $\mathbb{R}^{3\mathbb{Z}}$ where $\boldsymbol{U}_j = (\boldsymbol{U}_{j1}, \boldsymbol{U}_{j2}, \boldsymbol{U}_{j3})$:

$$\forall j \in \mathbb{Z} \quad \mathrm{curl}_h(\boldsymbol{U})_j = -\frac{1}{2|T_j|}\sum_{l\in\mathcal{N}_j} \boldsymbol{U}_l \times \boldsymbol{n}_{jl},$$

where $|T_j|$ denotes the area of the jth cell, $\mathcal{N}_j$ is the set of neighbours of T_j and $\boldsymbol{n}_{jl}$ is a vector of length $|F_{jl}|$ in the direction of the outer unit normal on the common face F_{jl} of T_j and T_l. For finite sequences $\boldsymbol{U}$ and $\boldsymbol{V}$, it may be shown that

$$(\mathrm{curl}_h(\boldsymbol{U}), \boldsymbol{V})_h = (\boldsymbol{U}, \mathrm{curl}_h(\boldsymbol{V}))_h, \tag{3}$$

where

$$(\boldsymbol{U}, \boldsymbol{V})_h = \sum_{j\in\mathbb{Z}} |T_j|\, \boldsymbol{U}_j \cdot \boldsymbol{V}_j$$

denotes the discrete scalar product.

The scheme that we are going to study is then defined as follows. Let $\boldsymbol{H}^{-1/2}$ and $\boldsymbol{E}^0$ be an approximation of the initial data, we define $\boldsymbol{H}^{n+1/2}$ and $\boldsymbol{E}^{n+1}$ by the recursive formula:

$$\left\{\begin{array}{lclcl} \boldsymbol{H}_j^{n+1/2} & = & \boldsymbol{H}_j^{n-1/2} & - & \Delta t \mu_j^{-1}\,\mathrm{curl}_h(\boldsymbol{E}^n)_j \\ \boldsymbol{E}_j^{n+1} & = & \boldsymbol{E}_j^n & + & \Delta t \varepsilon_j^{-1}\,\mathrm{curl}_h(\boldsymbol{H}^{n+1/2})_j \end{array}\right. \tag{4}$$

Let us briefly describe how the scheme (4) may be seen as a Finite Volume Scheme with a centered flux formula. To this end, we write the Maxwell equation in conservative form,

$$\frac{\partial \mathbf{Q}}{\partial t} + \mathrm{div}\,\mathbb{F}(\mathbf{Q}) = 0, \qquad \boldsymbol{Q}(\cdot,0) = \boldsymbol{Q}_0 \tag{5}$$

where $\mathbf{Q}$ is given by $\mathbf{Q} = {}^t(\boldsymbol{B}, \boldsymbol{D}) = {}^t(\mu\boldsymbol{H}, \varepsilon\boldsymbol{E})$ and the definition of $\mathbb{F} = (\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3})$ follows from (1).

In order to perform discretization in space, we choose the mesh elements as control volumes and integrate (5) over each cell T_j. The semi-discretized formulation then reads as follows,

$$\partial_t \mathbf{Q}_j + \frac{1}{|T_j|}\sum_{l\in\mathcal{N}_j} \boldsymbol{\Phi}_{jl}(\boldsymbol{Q}_j, \boldsymbol{Q}_l) = 0, \tag{6}$$

where $\boldsymbol{\Phi}_{jl} = \boldsymbol{\Phi}_{jl}(\boldsymbol{Q}_j, \boldsymbol{Q}_l)$ denotes the numerical flux across the interface F_{jl}. If we define the function $\boldsymbol{\Phi}_{jl}$ as the centered flux,

$$\boldsymbol{\Phi}_{jl}(\boldsymbol{U}, \boldsymbol{V}) = \frac{1}{2}(\mathbb{F}(\boldsymbol{U}) + \mathbb{F}(\boldsymbol{V}))\cdot \boldsymbol{n}_{jl}, \tag{7}$$

it may be easily seen that

$$\boldsymbol{\Phi}_{jl} = {}^t(\boldsymbol{\Phi}_{jl,\mathbf{B}}, \boldsymbol{\Phi}_{jl,\boldsymbol{D}}),$$

with

$$\frac{1}{|T_j|} \sum_{l \in \mathcal{N}_j} {}^t\left(\boldsymbol{\Phi}_{jl,\mathbf{B}}(\boldsymbol{D}_j, \boldsymbol{D}_l), \boldsymbol{\Phi}_{jl,\mathbf{D}}(\boldsymbol{B}_j, \boldsymbol{B}_l)\right) = {}^t\left(\mathrm{curl}_h(\varepsilon^{-1}\boldsymbol{D})_j, -\,\mathrm{curl}_h(\mu^{-1}\boldsymbol{B})_j\right).$$

Using an explicit second order leapfrog scheme for time discretization thus yields (4).

4. Stability

We aim to prove that the L^2-norms of the approximated fields are uniformly bounded with respect to the mesh parameter h and the time step n. This does recover partially the results in [4] where stability is proven for the same scheme on a bounded domain with respect to a so-called “Leapfrog”-energy which defines a positive quadratic form of all the unknowns under an appropriate CFL condition.

In order to get stability with respect to the L^2-norm, we introduce the L^2-like discrete electromagnetic energy

$$\mathcal{E}^n = (\varepsilon \boldsymbol{E}^n, \boldsymbol{E}^n)_h + (\mu \boldsymbol{H}^{n-1/2}, \boldsymbol{H}^{n-1/2})_h. \tag{8}$$

The CFL condition depends on the regularity of the mesh and the electromagnetic coefficients ε and μ. More precisely, let $\mathcal{T}_h$ be a grid of $\mathbb{R}^3$. We assume that any cell of the grid is an open, convex polyhedron with K faces. Further, let $C > 0$ and $c > 0$ be two constants such that

$$\begin{array}{ll} \forall j, l \in J & c\,h^2 \leq |F_{jl}| \leq C\,h^2, \\ \forall j \in J & c\,h^3 \leq |T_j| \leq C\,h^3. \end{array} \tag{9}$$

Let the time step Δt be chosen such that

$$\frac{1}{\alpha}\frac{KC}{4c}\frac{\Delta t}{h} \leq r < 1, \tag{10}$$

where $\alpha > 0$ is given by (2). Then

$$\mathcal{E}^n \leq \frac{1+r}{1-r}\mathcal{E}^0 \tag{11}$$

for all $n \in \mathbb{N}$.

5. Convergence Analysis on unstructured grids

The stability result of the previous section implies that a subsequence of $(\boldsymbol{H}_h, \boldsymbol{E}_h)_{h>0}$ converges weakly in L^2. The main result of this paper is the identification of the weak limit. It turns out that, in general, curl_h yields a nonconsistent discretization of the continuous curl operator. However, we are able to prove that the weak limit is a solution of (1) in the distributional sense if the meshes satisfy the following additional condition:

For any bounded domain $\Omega \subset \mathbb{R}^3$, we assume that

$$h \sum_{F_{jl}\cap\Omega\neq\emptyset} |\omega_j - 2\sigma_{jl} + \omega_l|^2 = o(1), \tag{12}$$

where ω_j denotes the mass center of the jth cell and σ_{jl} is the mass center of the common face of T_j and T_l.

Theorem 1 *Let $\mathcal{T}_h$ be a family of unstructured grids satisfying the regularity assumptions (9) and condition (12).*

Let Δt be chosen such that the CFL condition (10) is satisfied.

Then, the sequence $(\boldsymbol{H}_h, \boldsymbol{E}_h)_{h>0}$ given by scheme (4) from smooth initial data $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$, converges weakly in $L^2(\mathbb{R}^+, \mathbb{R}^3)^3$ to a solution $(\boldsymbol{H}, \boldsymbol{E})$ of the Maxwell system (1) in the distributional sense.

Let us sketch the main ideas of the proof. With a vector valued test function φ of $\mathcal{C}_0^\infty(\mathbb{R}^+ \times \mathbb{R}^3)^3$ we associate the sequence of $\mathbb{R}^{3\mathbb{Z}}$ defined by $\varphi_j^q = \varphi(t_q, \omega_j)$ where $t_q = q\Delta t$ for $q \geq 0$. We deduce from the definition of the scheme that

$$\sum_n \sum_j |T_j| \varepsilon_j (\boldsymbol{E}_j^{n+1} - \boldsymbol{E}_j^n) \cdot \varphi_j^n = \Delta t \sum_n \sum_j |T_j| \operatorname{curl}_h(\boldsymbol{H}^{n+1/2})_j \cdot \varphi_j^n. \tag{13}$$

Now, the left hand side of the above expression does correspond to the weak derivative in time of $\varepsilon \boldsymbol{E}_h$,

$$-\int_{\mathbb{R}^+\times\mathbb{R}^3} \varepsilon \boldsymbol{E}_h \cdot \partial_t \varphi \, d\boldsymbol{x}\, dt - \int_{\mathbb{R}^3} \varepsilon \boldsymbol{E}_0 \varphi(\cdot, 0) d\boldsymbol{x}$$

up to an additional correction term of order $\mathcal{O}(h)$. To prove this, we use similar techniques as in [8].

It remains to show that the right hand side of (13) yields the weak curl of $\boldsymbol{E}_h$. The crucial point in the proof is the following weak consistency of the discrete curl operator which is true under the additional condition (12) on the meshes:

Proposition 1 *Let $(\mathcal{T}_h)$ be a family of unstructured grids satisfying (9) and (12). Let $\varphi^h \in \mathbb{R}^{3\mathbb{Z}}$ the sequence defined by $\varphi_j^h = \varphi(\omega_j)$ for a given test function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$. Then*

$$(\mathrm{curl}_h(\boldsymbol{U}), \varphi^h)_h = (\boldsymbol{U}, (\mathrm{curl}\,\varphi)^h)_h + o(1) \tag{14}$$

for any finite sequence $\boldsymbol{U} = (\boldsymbol{U}_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{3\mathbb{Z}}$.

Let us finish with some remarks.

– The limit field $(\boldsymbol{H}, \boldsymbol{E})$ is of class L^2. Due to the linearity of the scheme and condition (12), it may be shown, however, that

$$(\boldsymbol{H}, \boldsymbol{E}) \in \mathcal{C}^0([0,T]; L^2(\mathbb{R}^3))^6.$$

– In §4, the discrete energy was shown to be bounded by the energy at the initial state up to a multiplicative factor depending on the CFL condition. Under the condition (12), we are able to prove that the discrete energy is conserved at the limit provided that the initial data are smooth:

$$\mathcal{E}^n = \mathcal{E}^0 + \mathcal{O}(h). \tag{15}$$

– The above-mentioned results finally imply the convergence of $(\|\boldsymbol{E}_h\|_0^2 + \|\boldsymbol{H}_h\|_0^2)^{1/2}$ to the L^2-norm of the limit field. Hence, the convergence in Theorem 1 is strong.

6. Numerical results in 2D

The numerical performance of the method has been illustrated in [10] with several examples. For instance, a comparison of the centered scheme (4) with a third order accurate MUSCL finite volume method clearly showed that (4) is not diffusive whereas the MUSCL method yields a bad approximation in a long time run. Figure **??** below represents the scattering of a monochromatic wave across the dielectrical layer of a coated airfoil (NACA0012). The simulation has been done with an unstructured grid corresponding to 15 points per wavelength, and we notice that no spurious oscillation occurs at the interfaces.

7. About the "aspect"condition

Let us finish with some remarks on condition (12). First, notice that for an arbitrary mesh, $|\omega_j - 2\sigma_{jl} + \omega_l|$ is of order $\mathcal{O}(h)$ such that the left hand side of (12) is uniformly bounded with respect to h. It evaluates to 0 on a structured grid. Condition (12) thus implies a certain homogeneity of the mesh. However, up to $\mathcal{O}(h^{-2})$ cell interfaces can violate (12) which allows, for instance, local mesh refinement near the border of the computational domain.

Condition (12) seems to be rather strong. The question is whether it is really necessary. Numerical tests on arbitrary generated meshes like the one in §6 have shown very satisfying results. Some meshes however, for which it may easily be seen that condition (12) does not hold, do not give good approximations of the electromagnetic field.

In order to illustrate these phenomena, we give below some numerical results in one dimension of space. Starting from sequences $u^0, v^{1/2} \in \mathbb{R}^{\mathbb{Z}}$ we define $u^n, v^{n+1/2}$ by

$$\left\{ \begin{array}{l} v^{n+1/2} - v^{n-1/2} = \Delta t \; D_h(u^n) \\ u^{n+1} - u^n = \Delta t \; D_h(v^{n+1/2}), \end{array} \right. \tag{16}$$

where, similarly to the discrete curl operator, the discrete differentiating operator D_h, acting on sequences of $\mathbb{R}^{\mathbb{Z}}$, is defined by

$$D_h(u)_i = \frac{u_{i+1} - u_{i-1}}{2h_i}$$

on a partition of $\mathbb{R}$ into cells C_i of (variable) size $h_i > 0$. In 1D, condition (12) reads as follows

$$\forall [a,b] \subset \mathbb{R} \quad \sum_{i : x_i \in [a,b]} h_i |r_i^h|^2 = o(1), \tag{17}$$

where

$$r_i^h = \frac{h_{i+1}}{h_i} + \frac{h_{i-1}}{h_i} - 2.$$

Let us define the "variation "of a grid of N cells by

$$\mathrm{var}(h) = \sum_{i=1}^{N} h_i |r_i^h|^2.$$

We tested the method on three grids with 200 cells, smooth initial data and periodic boundary conditions. Figures 1 and 2 represent the initial condition (solid line) and the approximation of u at $t = 0.6$ (dashed line) for different grids. The left picture of Figure 1 does correspond to a regular partition of constant path $h = 1/N$. Condition (17) is clearly satisfied, since $\mathrm{var}(h) = 0$. Next, we applied the method with an alternating sequence of cell sizes, i.e. $h_{2i-1} = h$ and $h_{2i} = 2h$, $\forall i \in \mathbb{N}^*$. The variation of the grid is evaluated to $\mathrm{var}(h) = 2$ and does not depend on the number of cells. The right picture of Figure 1 clearly shows that the method does not converge. The approximation seems to split into two functions with different amplitude. The two pictures of Figure 2 correspond to unstructured grids that have been generated using a random number generator. The grid with the smaller variation ($\mathrm{var}(h) = 0.016$) on the right clearly yields a better approximation.

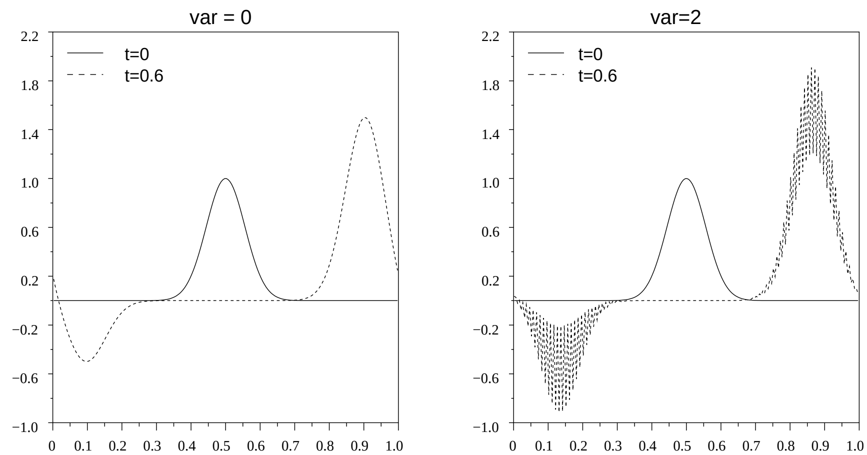

Figure 1. *Approximation of u with a regular and an alternating mesh.*

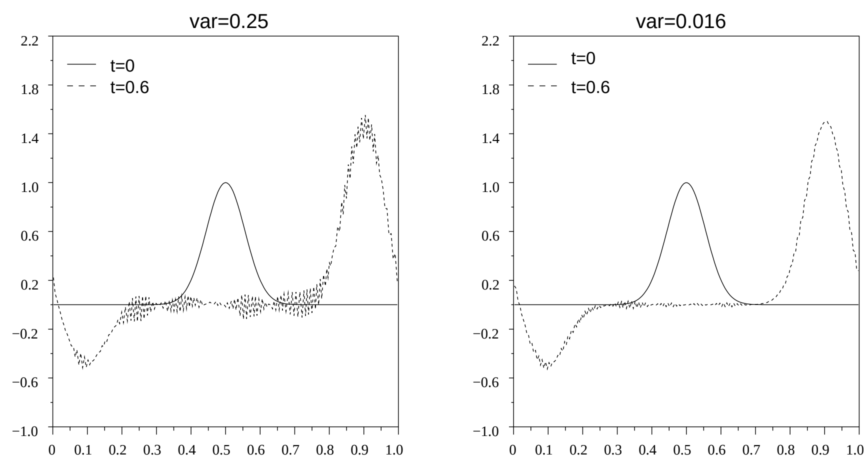

Figure 2. *Approximation of u with two random meshes of different variation.*

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